Chaos-hyperchaos transition

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The chaos-hyperchaos transition occurs when the second Lyapunov exponent becomes positive. We argue that this transition is mediated by changes in the stability of an infinite number of unstable periodic orbits embedded in the chaotic attractor. Bifurcations of unstable periodic orbits occur in the neighborhood of the chaos-hyperchaos transition point where we observe unstable variable dimensionality. We give evidence that the chaos-hyperchaos transition is initiated by (i) the saddle-repeller bifurcation of a particular unstable periodic orbit usually of low period, (ii) the appearance of a repelling node in the saddle-node bifurcation, after which the chaotic attractor becomes riddled, or (iii) the absorption of the repeller (unstable node or focus) originally located out of the attractor by the growing attractor.

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Extensive studies of the techniques to control [1] and synchronize [2] chaotic systems have stimulated growing interest in high-dimensional hyperchaotic systems. Such systems are characterized by at least two positive Lyapunov exponents for typical trajectories in the arbitrarily high phase space. The first example of such a system was presented by Rossler [3] for a model of a particular chemical reaction. Later, hyperchaotic attractors have been found in electronic circuits and other chemical reactions [4]. In [5] it was shown that by weakly coupling n chaotic systems it is possible to obtain a hyperchaotic attractor with n positive Lyapunov exponents. The transition from chaos to hyperchaos was studied in [6]. It was shown that at this transition the attractor's dimension and the second Lyapunov exponent grow continuously.

In this paper we consider a dynamical system evolving on a chaotic attractor A (i.e., with one positive Lyapunov exponent) and allow the control parameter to vary slowly in such a way that the second Lyapunov exponent becomes positive so that the attractor A becomes hyperchaotic. We give evidence that the chaos-hyperchaos transition is typically stretching (spreading) along the control parameter interval, and that its mechanism has the same characteristic features as the blowout bifurcation of attractors located at invariant manifolds in systems with symmetry [7]. We show that the transition to hyperchaos begins when a repeller arises in the attractor and that one of the following conditions should be fulfilled for this: (i) some usually low-periodic orbit γ, embedded in the chaotic attractor A undergoes a saddle-repeller bifurcation; (ii) a repelling node in the attractor appears in the saddle-node bifurcation; (iii) the repeller (unstable node or focus) originally located out of the attractor is absorbed by the growing attractor. At this control parameter point, the attractor loses its asymptotic stability and its basin becomes riddled (at least locally).

As an example, consider a two-dimensional map F in the form

$$\begin{aligned} x_{n+1}&=f_a(x_n)+\varepsilon[f_a(y_n)-f_a(x_n)],\\ y_{n+1}&=f_a(y_n)+\varepsilon[f_a(x_n)-f_a(y_n)], \end{aligned} \tag{1}$$

where $x_n, y_n \in \mathcal{R}$ are dynamical variables, $f_a(x)=ax(1-x)$ is a logistic map, a is the system parameter of the single map $f_a: x \mapsto f_a(x)$, and ε is the coefficient of coupling which we consider as a control parameter. The primary reason for choosing two-dimensional maps to illustrate our results is that, for such systems, there exists a procedure for computing unstable periodic orbits with higher periods embedded into a chaotic attractor with high precision [8].

Our map (1) is noninvertible and its attractor presents an invariant chaotic set which is enclosed in the so-called *chaotic area* (according to the terminology of Mira *et al.* [9]). This chaotic area is an invariant region of the two-dimensional phase space bounded typically by a finite number of segments of the so-called *critical lines*, which are obtained from the iterations of the set $L_0=\{(x,y): DF(x,y)=0\}$, i.e., curves where the Jacobian DF vanishes, where F is the map given by Eq. (1).

Examples of the attractors located inside the chaotic area, for $a=a_0\approx 3.678\,573\,51$ [10] and two different values of the coupling parameter ε are shown in Fig. 1(a,b). The attractor shown in Fig. 1(a) is a chaotic one characterized by the Lyapunov exponents $\lambda_1\approx 0.11$ and $\lambda_2\approx -0.04$, i.e., before the chaos-hyperchaos transition, while that in Fig. 1(b) is hyperchaotic with Lyapunov exponents $\lambda_1\approx 0.13$ and $\lambda_2\approx 0.0068$, i.e., after the transition. [It should be added that there exist other attractors symmetrical in relation to the invariant manifold $x=y$ to those shown in Fig. 1(a,b). These attractors have the same properties as those described below.] As we can see, the shapes of the attractors are similar, so the transition from chaos to hyperchaos seems to be soft and continuous as described in [6]. Although the system (1) has symmetry, its attractors shown in Fig. 1(a,b) are not located at the invariant manifold, so this system can be used for description of the general properties of the chaos-hyperchaos transition. Characterizing the chaos-hyperchaos transition in terms of periodic orbits embedded in the attractor, one observes that in the first, chaotic case, most of the periodic orbits in the attractor are saddles, while in the second case most of them are unstable nodes or focuses. Let us take into account the numbers of repellers (unstable nodes or focuses) and saddles inside the attractor A. These numbers up

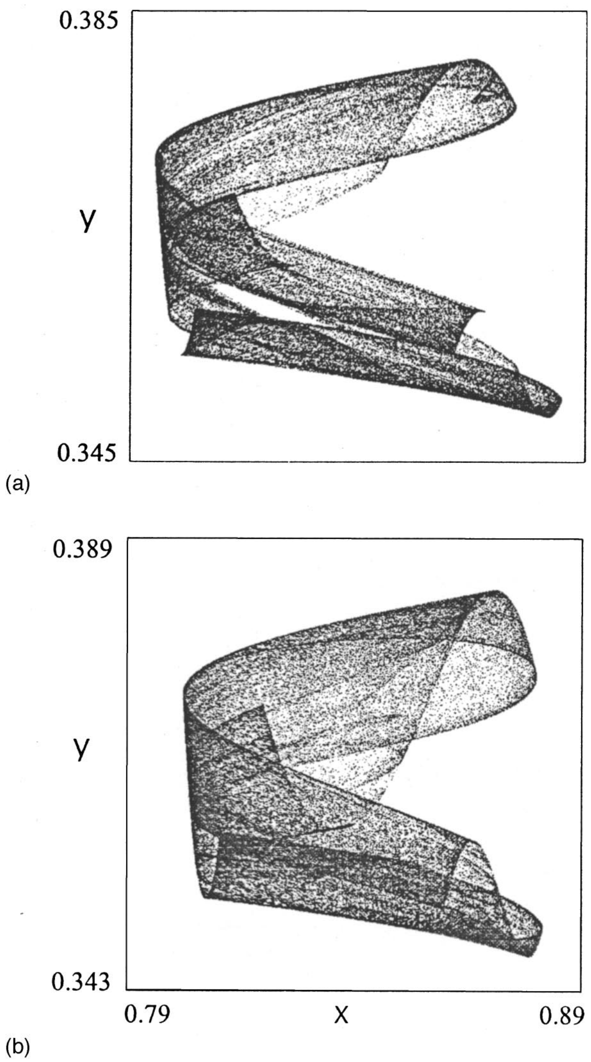

FIG. 1. Attractors of the map F for $a=a_0=3.678\,573\,51$; (a) chaotic at $\varepsilon=\varepsilon_1=0.964$, (b) hyperchaotic at $\varepsilon=\varepsilon_2=0.965$ (only one piece of four-piece attractors is shown).

to period 32 are collected in Table I (a,b) for $a=a_0$ and two different values ε_1 and ε_2 of the coupling parameter ε [corresponding to the attractors shown in Fig. 1(a,b)]. As can be seen, the number of saddles is greater than the number of nodes for $\varepsilon=\varepsilon_1$; and vice versa for $\varepsilon=\varepsilon_2$. This shows that for $\varepsilon=\varepsilon_1$ the attractor should be apparently chaotic, and at the value ε_2 hyperchaotic.

To be more accurate, we define the following quantitative characteristics for the chaos-hyperchaos transition: the saddle and repeller weights Λ_p^S and Λ_p^R of all period-p saddles and repellers. We will do it by analogy with the approach developed in [7] for the blowout bifurcation in systems with invariant subspace.

Let γ_i^p be a period-p orbit of the two-dimensional map F, and λ_1 and λ_2 its larger and smaller Lyapunov exponents, respectively, and let $\nu_1=e^{\lambda_1}$, $\nu_2=e^{\lambda_2}$ be the larger and smaller Lyapunov numbers. Then we define the following period-p saddle and repeller weights:

$$\Lambda_p^S=\sum_{i=1}^{N_p^S}\rho(\gamma_i^p,p)\lambda_2(\gamma_i^p,p),$$

TABLE I. Number of unstable periodic orbits of saddle and repeller types, $a=a_0=3.678\,573\,51$; (a) $\varepsilon=\varepsilon_1=0.964$; (b) $\varepsilon=\varepsilon_2=0.965$.

Period	Saddle	Repeller
	(a)	
4	0	0
8	1	1
12	0	0
16	2	0
20	3	1
24	4	0
28	4	2
32	4	1
	(b)	
4	0	1
8	0	2
12	0	2
16	2	2
20	1	5
24	3	5
28	4	6
32	3	10

$$\Lambda_p^R=\sum_{i=1}^{N_p^R}\rho(\gamma_i^p,p)\lambda_2(\gamma_i^p,p), \tag{2}$$

where N_p^S and N_p^R are the numbers of saddles and repellers, respectively; and the cycle γ_i^p weight

$$\rho(\gamma_i^p,p)=\frac{1/\nu_1}{\sum_{i=1}^{N_p^S+N_p^R}1/\nu_1}$$

approximates as $p\to\infty$ the natural measure of typical trajectories on A that stay close to the period-p orbit considered [11].

From the formulas for Λ_p^S and Λ_p^R one can find that at $a=a_0$: $\Lambda_{28}^S(\varepsilon_1)=0.66$, $\Lambda_{28}^R(\varepsilon_1)=0.0975$; $\Lambda_{32}^S(\varepsilon_1)=0.876\,88$, $\Lambda_{32}^R(\varepsilon_1)=0.0648$; and $\Lambda_{28}^S(\varepsilon_2)=0.514\,154$, $\Lambda_{28}^R(\varepsilon_2)=4.774\,911\,5$; $\Lambda_{32}^S(\varepsilon_2)=0.066\,985$, $\Lambda_{32}^R(\varepsilon_2)=3.557\,403\,2$.

These calculations also support the fact that at $\varepsilon=\varepsilon_1$ the attractor A is chaotic, but at $\varepsilon=\varepsilon_2$ hyperchaotic. They also give evidence that the transition from chaos to hyperchaos occurs when the numbers of saddles and repellers in the attractor are balanced, which can be indicated by approximately the same saddle and repeller weights, for a sufficiently large period p. At the bifurcation point, the second Lyapunov exponent $\lambda_2=\lambda_2(A)$ of the attractor A should be equal to zero; it can be found that the parameter $\varepsilon=\varepsilon_b$ value for this is $\varepsilon_b=0.964\,995$ (for $a=a_0$).

The critical value $\varepsilon=\varepsilon_b$ when the transition from chaos to hyperchaos occurs is analogous to the point of blowout bifurcation in systems with invariant subspace [7], and we come to the conclusion that the chaotic attractor becomes hyperchaotic due to the same mechanism as a blowout bifurcation.

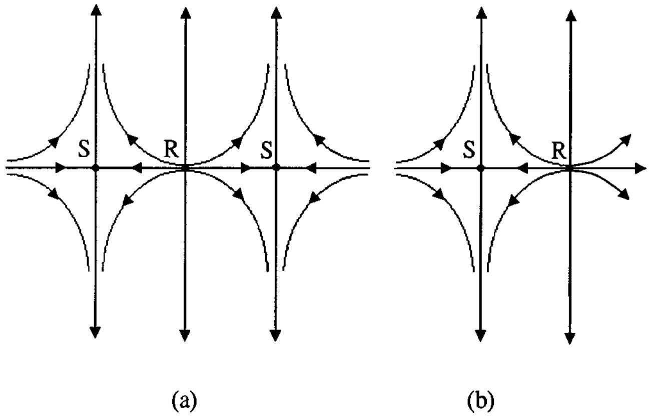

FIG. 2. Bifurcations leading to the occurrence of repellers inside chaotic attractors.

Varying the coupling parameter further to the right we observe that the relative number of repellers inside A increases more and more. Finally, at some value $\varepsilon=\varepsilon_s$ the last saddle disappears, so that for $\varepsilon>\varepsilon_s$ all unstable periodic orbits inside the attractor are repellers.

Consider the bifurcation in which the first periodic orbit γ inside the attractor becomes a repeller. Three possibilities for this can be realized in our model. (1) It is due to a saddle-repeller bifurcation: the eigenvalue $\nu_2(\gamma)$ (corresponding to a stable direction of the second saddle cycle γ) crosses -1; the saddle becomes an unstable node creating a saddle of doubled period [Fig. 2(a)]. (2) It is due to a saddle-node bifurcation: a saddle and unstable node are created; the second eigenvalue $\nu_2(\gamma)$ is equal to 1 at the moment of bifurcation and then becomes larger than 1 [Fig. 3(b)]. (3) It is due to the absorption (hunting) of the repeller (unstable node or focus), situated at some rather small distance from the attractor, by the growing attractor.

Schematic representations of the behavior of the trajectories in the neighborhood of the bifurcating cycles in the first two cases are shown in Fig. 2(a,b). They give evidence that there exist trajectories which, when starting arbitrarily close to the repeller R, approach the saddle at some sufficiently small distance and later go along the unstable manifold. Mechanisms (1) and (2) for the appearance of the repeller in the chaotic attractor can be considered as inner scenarios and the third one as an outer one.

Suppose that (1) or (2) takes place and the newly created repeller R belongs to the attractor, but the saddles do not. This supposition describes one of the typical situations. Indeed, if our attractor A is still chaotic (not hyperchaotic), it cannot cover the whole open region in R^2 phase space. In this case, one of the possibilities is that A has a ‘‘Cantor structure’’ in the direction transverse to the ‘‘sheets’’ of the attractor [9]. If this holds, we have a repelling periodic orbit R connected by a separatrix with the saddle S, which does not belong to the attractor, together with some of its neighborhood.

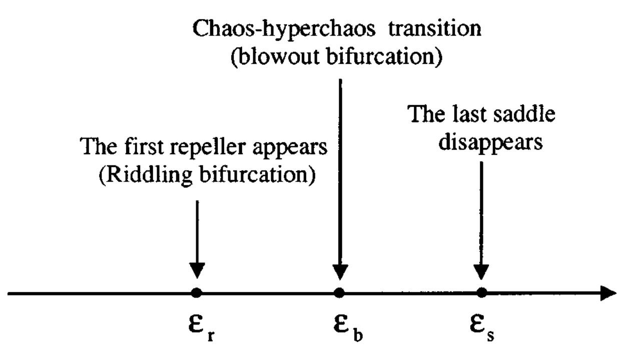

FIG. 3. Control parameter interval of the chaos-hyperchaos transition.

For a class of two-dimensional noninvertible maps like Eq. (1) this implies that arbitrarily close to the repeller R (belonging to the attractor A) one can find [9] a positive measure set of points which after some iterations will enter a sufficiently small neighborhood of S (they are in the tongues issuing from R and bounded by the invariant curves of the map F). Consequently, these points will go away from the attractor and just after the bifurcation the attractor A cannot be asymptotically (topologically) stable any longer but only stable in the Milnor sense [12].

The situation is analogous when both the repeller R and the saddle S belong to the attractor. Indeed, in this case, the separatrix connecting the repeller R with the saddle S should cross (contain) intervals of points not belonging to the attractor A [9]. Preimages of such intervals, those belonging to the separatrix and not belonging to A, will be accumulating in the repeller R (more accurately, some part of the preimages will have this property so that our map F is noninvertible). Therefore, by the same arguments, there exists a positive measure set of points in the neighborhood of the repeller R which will go away from the attractor (at some distance corresponding to the scale of the intervals mentioned).

Note that both the repeller and the saddle(s) are situated inside the shape of the chaotic area which is bounded by critical lines. The trajectories that first go out from the attractor (by the mechanism described above) cannot escape from the chaotic area. Finally, almost all of them will be attracted back by the chaotic attractor A if there are no other attractors inside the chaotic area. In this case, the basin of attraction of the attractor A will be locally riddled [13]. To be convinced of this, consider a set of preimages of the repeller R. As our map F is noninvertible [each point (x,y) has 0, 2, or 4 preimages], and as numerical evidence supports, the set of preimages of R is dense in the attractor A [computations using analytical formulas for the inverse map show that the preimages of R create the same shape as the attractors shown in Fig. 1(a,b)]. We cannot prove this fact analytically as our map (1) is two dimensional, but for one-dimensional maps this property is rigorously proved for attractors with a ‘‘good’’ invariant measure [14]. Now, if the set of preimages is supposed to be dense in the attractor A, we can easily come to the conclusion that after the bifurcation of the creation of the repeller inside A, the basin of attraction of A is riddled [15] at least locally [16].

This basin may also be globally riddled if there exist other attractors inside the shape of the chaotic area. The existence of such attractors (usually they develop from attracting cycles with a narrow radius of attraction) is rather typical for two dimensional (discrete) maps. It may be related to the existence of so-called Newhouse regions in parameter space, caused by the persistence of homoclinic tendencies [17]. For a Hénon map, the phenomenon was shown to be typical in [18].

Let us fix the system parameter $a=a_0$ and choose the coupling parameter ε such that A will be a chaotic attractor and all unstable periodic orbits embedded on it are saddle points. Then varying (increasing in our case) ε we find the moment of the appearance of the first repeller R inside A. It appears to happen in accordance with the mechanism (1): a period-8 saddle transforms to an unstable node in a supercritical period doubling, at $\varepsilon=\varepsilon_r\approx 0.9277$ [Fig. 2(a)] and after this bifurcation the basin of A should become riddled. With a further increase of ε, saddle-repeller bifurcations of the same type for two period-28 cycles are observed. After them, a period-20 repeller arises in a saddle-node bifurcation [mechanism (2) as in Fig. 2(b)]. With further increase of ε, more and more unstable nodes appear as a result of all three mechanisms: period-doubling, saddle-node, and absorption bifurcations. Some of the unstable periodic orbits can transform into unstable focuses in a Hopf bifurcation.

The first bifurcation in which the first periodic repeller appears in the chaotic attractor A at $\varepsilon=\varepsilon_r$ may be called the *riddling bifurcation*—analogously to the same type of bifurcation known for systems with the chaotic attractor in invariant subspace [15]. Therefore, from our observations, we may conclude that the transition from a chaotic to a hyperchaotic attractor starts from a riddling bifurcation. Therefore, we conclude that the chaos-hyperchaos transition is accompanied by the existence of a control parameter interval $(\varepsilon_r;\varepsilon_s)$, in which one observes unstable variable dimensionality, shown in Fig. 3. The following conditions apply. (i) For $\varepsilon<\varepsilon_r$ all unstable periodic orbits embedded in the chaotic attractor are of the saddle type. (ii) $\varepsilon=\varepsilon_r$ is the control parameter value (of the riddling bifurcation) at which the first repeller appears in the attractor. (iii) In the interval $(\varepsilon_r,\varepsilon_b)$ infinitely many unstable periodic orbits bifurcate from the saddle to the repeller and new repellers appear in the chaotic attractor. (iv) The transition chaos-hyperchaos occurs at some parameter value $\varepsilon=\varepsilon_b$ inside the interval $(\varepsilon_r,\varepsilon_s)$; at this parameter value the weights of saddles and repellers [defined by Eq. (2)] are balanced. (v) $\varepsilon=\varepsilon_s$ is the control parameter value at which the last saddle disappears and for $\varepsilon>\varepsilon_s$ all unstable periodic orbits embedded in the chaotic attractor are repellers.

When the chaotic attractor A of a two-dimensional map located at the invariant manifold undergoes a blowout bifurcation it is replaced by (i) an attractor at infinity, or (ii) a chaotic or (iii) hyperchaotic attractor located out of the invariant manifold, or (iv) periodic attractor. In cases (ii) and (iii) we observe a sudden increase of the attractor's dimension from a value smaller than or equal to 1 to a value significantly larger than 1 (but smaller than or equal to 2). This phenomenon is connected with the sudden increase of the dimension of the manifold on which the system evolves before and after the blowout bifurcation rather than with the bifurcation mechanism. At the chaos-hyperchaos transition we do not observe a sudden increase of the attractor's dimension as in [6]. Other features of the blowout bifurcation, like the riddled basins (at least locally riddled [19]) and the changes in the stability of an infinite number of unstable periodic orbits embedded in the chaotic attractor, are present in the chaos-hyperchaos transition, so we have strong evidence that the mechanism of this transition is the same as the mechanism of the blowout bifurcation.

To summarize, we have shown here that the transition from chaos to hyperchaos is a bifurcation that, like a blowout bifurcation in a system with symmetry, is mediated by an infinite number of unstable periodic orbits. Infinitely many unstable periodic orbits become repellers in the neighborhood of this transition. The whole process is initiated when one of the following conditions is fulfilled: (i) the saddle-repeller bifurcation of a particular unstable periodic orbit usually of low period, (ii) the appearance of a repelling node in the saddle-node bifurcation, after which the chaotic attractor becomes riddled, or (iii) the absorption of the repeller (unstable node or focus) originally located out of the attractor by the growing attractor, which leads to the occurrence of a riddled basin of the chaotic attractor.

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